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NUMERICAL MODELLING FOR CLAYS WITH INTERNAL DISCONTINUITY SURFACES

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SUMMARY

A numerical model capable of performing deformation analysis of a medium containing discontinuity surfaces is presented. The discontinuity can be either a crack, which can be open or closed, or a shear band. The model consists of two separate numerical algorithms, which are coupled together by means of the principle of superposition. In particular, an integral equation scheme based on the theory of dislocations is adopted for modelling the discontinuity, while a finite element discretization is used for the continuous medium. In this paper the discontinuity modelling is illustrated in detail, together with the specific formulation of the principle of superposition adopted, and some simple examples of application are presented. The well-known modelling approach based on Fracture Mechanics theory is also briefly discussed. The two models are compared and some advantages and drawbacks of each are pointed out, comments are made regarding their applicability in the specific case of soil mechanics, and conclusions are drawn as regards the conditions under which one or the other is appropriate. Finally, a full-scale example of deformation analysis using the proposed model is presented. Copyright © 1999 John Wiley & Sons, Ltd.

Key words: discontinuity; deformation analysis; overconsolidated clays; constitutive laws

1. INTRODUCTION

Numerical modelling for the behaviour of clays and soils in general is becoming extremely sophisticated. This is an understandable result of the actual very complex behaviour of the material itself, together with the enhanced possibilities of today's computers. There is no longer the need to fall back upon linear elasticity as a constitutive law in deformation analyses, for the sake of simplicity. The possibility to contemplate complex constitutive relations in order to describe stress–strain behaviour has promoted the development of soil models which use a great number of parameters.

When a constitutive model is successful in representing the behaviour of a particular material, the problem arises on how to apply this knowledge to real situations; for example, such matters become important as how boundary conditions should be formulated, how to deal with soil non-homogeneity, etc.

Stiff overconsolidated clays containing discontinuity surfaces are typical examples of such situations. In fact, the *in situ* stress–strain response of such materials often presents aspects not possible to detect in laboratory testing on small samples. The reason for this apparent discrepancy is clear considering that for the *in situ* situation, where the soil mass contains discontinuity surfaces, it is often the behaviour of these which determines the global behaviour.

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Numerical modelling for overconsolidated clays containing discontinuites inevitably reflects these difficulties. Existing numerical methods are often stretched to extreme limits in order to make a model applicable to complex real-life situations. Finite element meshes for example need to be extremely refined in order to capture the presence of a crack in the medium; this can often be the source of numerical problems. If in such a situation the Finite Element method must also take account of the intrinsic complexity of the material, via a constitutive model which for example includes softening, numerical solutions present a formidable task and in fact remain tools bound to research use, rather than actual design.

Several attempts can be found in the literature where a Fracture Mechanics approach is used in order to model the behaviour of a soil mass containing discontinuities.[1–5] Such methods are primarily concerned with the behaviour of the discontinuity itself, i.e. if and when it will propagate, rather than with the overall response of the soil mass. Finite Element modelling in this context requires a very fine mesh in the vicinity of the discontinuity, and remeshing at every slight variation of its geometry, and therefore appears excessively cumbersome. Furthermore, in geotechnical engineering the global behaviour of the soil mass is often of greater interest than a precise knowledge of how the discontinuities develop.

An alternative method is available, which is based on the theory of dislocations.[5] In such cases the discontinuity surface is modelled as a continuous distribution of point dislocations. Attention is still focused on the behaviour of the discontinuity, but the stress–strain response at any point of the soil mass can also be derived.

The present work is motivated by the need for a reliable tool for modelling *in situ* behaviour of clayey soils which contain discontinuities, with particular reference to those cases where, while constitutive behaviour of the material is fairly well understood, it is not of much help in modelling global behaviour. The presence of some kind of discontinuity in the soil mass, and its exact position, are assumed to be known *a priori*. The Fracture Mechanics approach is first considered, and some comments are made regarding its applicability in Soil Mechanics problems. The displacement discontinuity method is then examined in detail, and a numerical model is presented which is capable of handling correctly, and with minimum computational expense, some of the above problems. The idea behind the model is that when a discontinuity is present inside a clayey soil it is this that will control the overall behaviour, acting as a kinematic constraint, while constitutive behaviour of the soil element plays a lesser role. In fact, concentration of strains over a small area (i.e. the discontinuity) leaves most part of the soil just slightly strained, and elastic behaviour could be considered for modelling those areas. This in part adheres to the philosopy of Fracture Mechanics, although in that case there is no theoretical provision regarding overall deformation analysis. The aim of the present work is to link a model which describes the behaviour of the discontinuity locally to the effects this behaviour produces in the soil mass; in this way a global model is formulated, which describes local and overall behaviour and the interaction between the two.

The final result is a 'hybrid' numerical model,[6] in which discontinuities are modelled via a surface integral method using the known influence functions for point dislocations in an elastic medium; the equations to be solved are comparable to those of the indirect approach of the Boundary Element method.[7,8] The soil mass on the other hand is modelled using the Finite Element method and assuming linear elastic behaviour. This can be extremely advantageous, as any changes in the discontinuity geometry (due, for example, to its propagation or to changes of position for parametric analysis) leaves the finite element mesh unaltered. Another advantage is that there is no need for mesh refinements in order to model the presence of a discontinuity;

a small number of elements is therefore sufficient to model even large-scale problems. The two formulations are coupled together using the principle of superposition; this is possible, since the material remains within the range of applicability of the principles of linear elastic theory.

The model here presented differs from other existing methods in various aspects. Firstly, any geometry and any number of discontinuities can be accounted for. The method is not a theoretical speculation on the local behaviour of discontinuities as in Fracture Mechanics, but attention is focused on modelling the interaction between discontinuity and continuum and on coupling these two aspects of behaviour. It is thus possible to make computations of how the global behaviour of a continuum is altered by the presence of a discontinuity.

Some important conlcusions are drawn regarding the applicability of the various modelling strategies for fractured clays. It is shown how it can be ultimately misleading to attempt a separation between 'correct' and 'not-correct' approaches; on the other hand, it is useful to be able to distinguish, on a rational basis, those situations where one or the other of the modelling strategies should be best applied.

In the next section the basic principles of Fracture Mechanics are briefly reviewed; particular reference is made to the assumptions which tacitly lie behind the derivation of the most commonly used equations. Then, in Section 3, the basic formulation of the displacement discontinuity model is illustrated, as adjusted in order to adapt it to soil. Stress intensity factors can still be calculated, but a comparison with those resulting from Fracture Mechanics theory is not always possible, owing to that fact that stress can be transmitted across the discontinuity surfaces. The accuracy and competency of the model are illustrated in the same section in an example of application. Finally, in Section 4 some considerations regarding the conditions for correct use of different modelling strategies are advanced. In Section 5 a full-scale application to the case of a well-known excavation is presented and the ability of the model to calculate global displacements is illustrated.

2. THE FRACTURE MECHANICS APPROACH

In Fracture Mechanics the aim is to be able to predict the stress states which would produce a fracture propagation process within a continuous medium, so that a prefixed safety margin may be prescribed. Fracture Mechanics can therefore be thought of as a complement to plasticity theory for the design of structures.

The basic equations of Fracture Mechanics are derived within the framework of linear elasticity theory. Expressions for the stress state in the small region around the fracture's tip can be found; these, for a straight crack of length $2a$ in a stressed infinite medium (Figure 1), are given by

$$\begin{aligned}
\sigma_x &= \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\\
\sigma_y &= \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\\
\tau_{xy} &= \frac{\sigma\sqrt{\pi a}}{\sqrt{2\pi r}} \sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}
\end{aligned} \tag{1}$$

where r and θ are polar co-ordinates with the crack tip as an origin.[9–11,14]

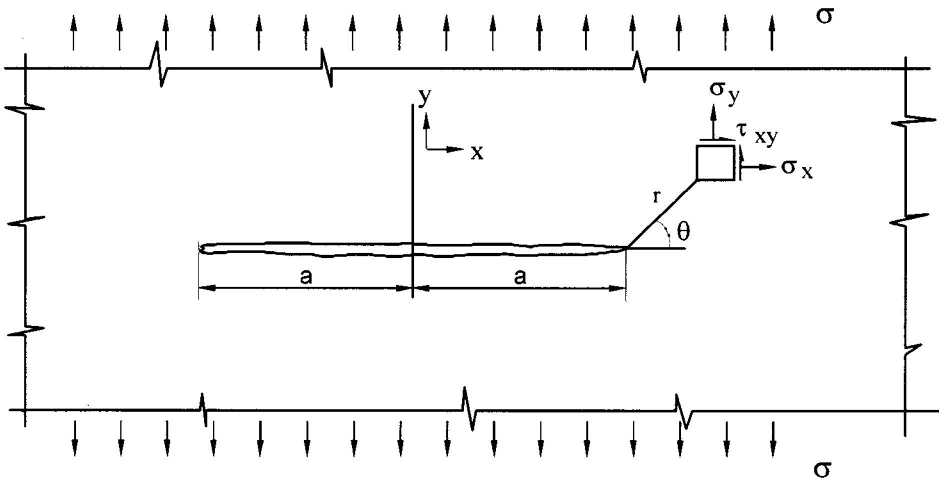

Figure 1. Straight crack in a stressed infinite medium

It is important to note that these are universal expressions, and in particular they do not depend upon material characteristics. The stress state around the tip is therefore only determined by geometry (position of the point at which stress is calculated, with respect to the tip), and is proportional to a parameter K called the stress intensity factor. In the simple case here examined of an opening crack in an infinite medium, the stress intensity factor is given by $K = \sigma\sqrt{\pi a}$.

Furthermore, considering that, according to equation (1), the stress state tends to infinity for $r \to 0$, i.e. very near the crack tip, the stress intensity factor can be assumed as a measure of the degree of singularity of the stress state.

Particular attention should be drawn to three facts:

(1) one of the boundary conditions employed in the derivation of equation (1) is that the crack surfaces are stress free, as in fact is the case for an opening tension crack;
(2) equation (1) is actually only the leading term of a series expansion of the exact mathematical stress field solution; the other terms can be neglected for points at a very small distance from the crack tip, i.e. when r is very small with respect to the crack half-length a;
(3) in a small zone around the crack tip the stress state is such that yielding may occur. Adopting a suitable yield criterion it is possible to estimate the extent of this zone by detecting where this criterion is overpassed; the 'plastic' zone can thus be shown to be actually very small. However, this calculation is based on the same cut-off in the series expansion mentioned above, so that, if the plastic zone is not small to start with, the procedure has no meaning.

One of the principal aims of Fracture Mechanics is to establish the crack propagation criterion, having unequivocally associated the physical phenomena of crack propagation with a process of separation.[10,11] It is clear that propagation will ensue when the stress state at the tip reaches some critical value; the parameter K hence offers a possible means of comparison between different situations, and a propagation criterion can be formulated on the basis of the value assumed by the stress intensity factor. It should be noted however that such a comparison based on K is

meaningful only as long as the truncation of the series expansion leading to equation (1) remains valid.

In practice, application of a propagation criterion based on the stress intensity factor consists in comparing the value assumed by the latter with a critical value K_C, which correpsonds to that value of the stress intensity factor at which a crack will propagate, and is therefore a material constant. In other words, K_C is the stress intensity factor characterizing the stress state at which, in a given material, fracture propagation ensues; its value is independent of the current configuration, both in terms of applied stresses and crack geometry. The physical meaning of K_C is therefore similar to that of the yield stress of a given material, which characterizes the stress state when yielding occurs. It is not superfluous to recall at this point that it is a basic concept of Continuum Mechanics that yielding or plastic collapse indicate material failure or plastic deformation which takes place by slip or shear deformation; even when a yield criterion such as the Tresca criterion is used, where the collapse condition is given as a relationship between normal stresses, the actual physical phenomenon at the grain scale involved is one of shear, and is related to the shear stresses. Plastic deformation will not take place unless the shear stress is sufficient to cause slip along some surface. In the case of fracture propagation on the other hand, the physical process is one of separation or opening, at molecular or grain scale; the mathematical model in fact describes just such a phenomenon. The material constant which quantifies the process indirectly supplies information regarding the molecular or crystalline structure of the material. The strains involved are normal tensile strains, and crack propagation will only take place when these are sufficiently high to cause grain separation, or opening.

In relation to these considerations, three modes of loading for an existing fracture are defined in Fracture Mechanics:[10,11] opening (Mode I), slip (Mode II) and out-of-plane (Mode III). Of course, the last does not regard plane problems, while the first is of primary importance in applications. In fact, though a fracture can be loaded (and hence propagate) by a combination of modes I and II (the so-called mixed mode conditions), the cases in which a pure Mode II condition is present are indeed very rare, and in fact non-existent except in very particular laboratory experiments.

3. THE DISPLACEMENT DISCONTINUITY MODEL

3.1. Introduction

In this section a complete numerical model for analysing the behaviour of an elastic medium containing one or more discontinuity surfaces is presented. The model is based on a combination of two separate formulations: of these, one, derived from the displacement discontinuity method, deals with the modelling of a discontinuity, while the other consists in a straightforward application of the finite element method to model a continuous medium. These two formulations are coupled by means of a numerical technique based on the principle of superposition.

In the displacement discontinuity method the concept of point dislocation is taken up and the theory of dislocations applied, so that a displacement discontinuity of finite size can be shown to be mathematically equivalent to an array of continuously distributed point dislocations. The presence of a shear band or of an opening crack may thus be modelled as a continuous distribution of such point dislocations; the correctness of such an approach from a mathematical point of view is rigorously demonstrated in the literature.[12]

This method leads to analytical expressions for the global stress and strain fields within a medium, due to the presence of such a discontinuity; a singular integral equation arises, for which a numerical procedure that leads to an accurate solution is available. In the vicinity of the crack tip, and in the cases where the basic assumption of Fracture Mechanics is satisfied (i.e. stress-free crack surfaces), the method yields a stress field which is identical to that of equation (1). Furthermore, the method allows a very simple computation of the stress intensity factors based directly on their definition as measures of the stress field singularity at the tip of the discontinuity. An example is provided in which results obtained through the presented algorithm are validated against known Fracture Mechanics results.

In the present paper the term 'discontinuity' will be used indiscriminately to indicate an internal surface across which relative displacements can occur. This surface may be physically present, and therefore can be thought of as an internal boundary surface across which there is a jump in the displacement field, or may be an ideal surface, as would occur across a region of localized shear straining. The relative displacement across such surfaces can be an opening or slip (that is, with the relative displacement vector perpendicular or parallel to the discontinuity surface, respectively), or, in the general case, a combination of the two.

3.2. Displacement discontinuity in an infinite medium

In dislocation theory[12] a point dislocation is quantified by means of a vector, called the Burgers vector and is defined as (Figure 2):

$$b_i = \int_\Gamma \frac{\partial u_i}{\partial x_j}\,\mathrm{d}x_j \tag{2}$$

where Γ is a circuit enclosing the dislocation and $\mathbf{u}(\mathbf{x})$ is the elastic displacement field.

The physical meaning attached to this vector is that of describing the relative displacement which occurred between two material points that were originally coincident at the same geometrical point.

A displacement discontinuity such as a crack cannot however be adequately described by a series of Burgers vectors, since a discontinuous function would arise. This obstacle is circumvented by introducing the concept of dislocation density, and thus transforming the

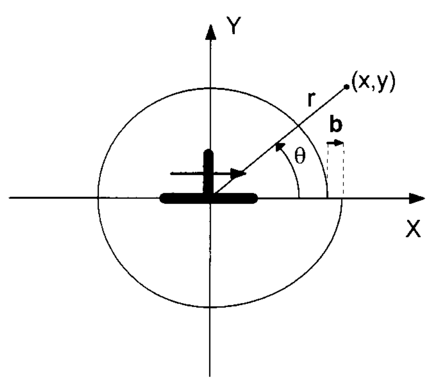

Figure 2. Point dislocation in the X direction at the origin of Cartesian axes

unknown function into a continuous one. A function $\delta(\mathbf{x})$ is therefore defined, such that

$$\mathrm{d}\mathbf{b} = \delta(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{3}$$

where $\delta(\mathbf{x})$ is the number of point dislocations with unit Burgers vector present in unit length. The quantity $\mathrm{d}\mathbf{b}$ therefore expresses the relative displacement at a point $\mathbf{x}$ and is mathematically equivalent to the Burgers vector of a fictitious point dislocation present at $\mathbf{x}$.

Linear elastic continuum mechanics theory provides expressions of influence functions to calculate the stress field at point $\mathbf{x}$ due to a generic point dislocation at $\mathbf{x}'$. These influence functions $\Psi(\mathbf{x}, \mathbf{x}')$ depend on two constants describing material properties, for example, Young's modulus and Poisson's ratio, and present a singularity of the order $1/r$, where r is the distance between the two points $\mathbf{x}$ and $\mathbf{x}'$. Considering an infinite medium in which an arbitrary number M of discontinuity surfaces is present (Figure 3), the state of stress arising at a generic point $\mathbf{x}$ of the medium as a consequence of the presence of the discontinuities can be obtained by integrating equation (3) along all the discontinuity surfaces

$$\sigma_\alpha(\mathbf{x}) = \sum_{m=1}^{M} \int_{\Gamma_m} \Psi_{\sigma_\alpha}^{\delta_\beta}(\mathbf{x}, \mathbf{x}')\,\delta_\beta(\mathbf{x}')\,\mathrm{d}\Gamma_m \tag{4}$$

where $\Psi_{\sigma_\alpha}^{\delta_\beta}(\mathbf{x}, \mathbf{x}')$ is the influence function by means of which the αth stress component at point $\mathbf{x}$ ($\sigma_\alpha(\mathbf{x})$) due to a unit βth dislocation component ($\delta_\beta(\mathbf{x}')$) present at point $\mathbf{x}'$ can be computed. For a two-dimensional problem, index α assumes values s(hear) and n(ormal), while the point dislocations can have components in the global X and Y directions; Γ_m is the mth discontinuity surface.

If the geometry of the various discontinuity surfaces is known, the stress field at any point of the continuous medium can be computed. Displacements can be calculated instead of stresses in a similar fashion, using appropriate influence functions.

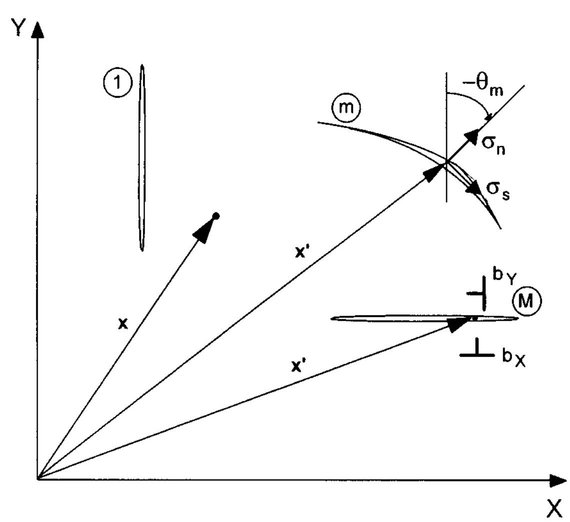

Figure 3. Plane system of M discontinuities

Such calculations however can only be performed after the dislocation density function $\delta(\mathbf{x})$ is known. On the other hand, equation (4) can also be employed to calculate this function, provided it is appropriately written with reference to chosen points at which the stresses $\sigma_\alpha(\mathbf{x})$ are known; such points are for example all points lying along the discontinuity surfaces. In this fashion, equation (4) becomes an integral equation in which the function $\delta(\mathbf{x})$ is the unknown; the equation is singular as a consequence of two facts, namely the already-mentioned singularity of order $1/r$ of the Ψ functions and because the unknown function $\delta(\mathbf{x})$ tends to infinity when the tip of the discontinuity is approached. This last singularity is of $r^{-1/2}$ order.[12]

It is possible to obtain a solution for this singular integral equation using standard collocation methods. However, a particular collocation method is available, through which a solution that is analytically exact[13] at certain collocation points is obtained; the method is based on Chebyshev polynomials and leads to a set of collocation equations, through which the unknown function f is computed at specific collocation points. To this end the unknown function δ has to be substituted by a function $f(\mathbf{x})$ defined as

$$\delta_\beta(s) = \frac{f_\beta(s)}{\sqrt{1-s^2}} \tag{5}$$

where s is a dimensionless curvilinear co-ordinate, appropriately taken along the discontinuity surface. This substitution is theoretically rigorous when the medium is homogeneous, and it results in the unknown function being non-singular while the known singularity of the dislocation density function $\delta(\mathbf{x})$ is respected.[5]

The singular integral equation to be solved is therefore transformed into

$$\sigma_\alpha(s) = \int_{-1}^{1} \Psi_{\sigma_\alpha}^{\delta_\beta}(s, s')\ \frac{f_\beta(s')\,\mathrm{d}s'}{\sqrt{1-s'^2}} \tag{6}$$

where appropriate rotations of co-ordinate axes using Mohr circle transformations have been performed; this integral equation can now be solved using the already-mentioned collocation method.

Combining the set of collocation equations with the appropriate boundary conditions,[9,5,6] the problem of a displacement discontinuity in an infinite medium can therefore be transformed into an algebraic system of linear equations

$$\begin{Bmatrix} \sigma_n(s_r) \\ \sigma_s(s_r) \end{Bmatrix} = \sum_{m=1}^{M} \frac{\pi}{N} \sum_{k=1}^{N} \begin{bmatrix} \Psi_{\sigma_n}^{\delta_x}(s_r, s_k) & \Psi_{\sigma_n}^{\delta_y}(s_r, s_k) \\ \Psi_{\sigma_s}^{\delta_x}(s_r, s_k) & \Psi_{\sigma_s}^{\delta_y}(s_r, s_k) \end{bmatrix} \cdot \begin{Bmatrix} f_x(s_k) \\ f_y(s_k) \end{Bmatrix} \quad (r = 1, 2, \ldots, N-1) \tag{7}$$

where s_k, s_r are the zero points of Chebyshev polynomials of the first and second species, respectively. Equation (7) can be written concisely as

$$\mathbf{T} = \mathbf{C} f \tag{8}$$

which can be easily solved using any available technique.

Regarding vector **T**, whose terms represent known values of the stress components along the discontinuity surface, normal and shear stress components are assumed as follows:

$$\sigma_n(s) = \begin{cases} 0 & \text{(opening crack—no internal pressure)} \\ p & \text{(opening crack—internal pressure} = p) \\ \sigma_n(\mathbf{x}) & \text{(no opening—field solution compressive normal stress)} \end{cases}$$
$$\sigma_s(s) = \begin{cases} 0 & \text{(opening crack)} \\ 0 & \text{(sliding discontinuity—no shear transmitted)} \\ \tau(\mathbf{x}) & \text{(sliding discontinuity with shear strength)} \end{cases} \tag{9}$$

The value of $\tau(\mathbf{x})$ depends on material characteristics, and will not necessarily be a constant. For example, it may depend on the relative displacement actually occurring along the discontinuity surface,[1] thus making the problem non-linear. In the present work, and for the sake of simplicity, constant values have been assumed for the shear stress components along the discontinuity surface; these have the physical meaning of a residual shear strength of the material, the assumption being appropriate for total stress analyses of clayey soils. When this shear stress component is assumed to be zero, the behaviour of the discontinuity is equivalent to what in Fracture Mechanics is commonly referred to as 'pure Mode II loading', but with the complete transmission of compressive stress components across the discontinuity surface correctly taken into account.[15]

Solution of equation (8) yields values of the dislocation density function $\delta(\mathbf{x})$ at the prescribed collocation points along the discontinuity surface. Once these are known it is possible to calculate, by integration of equation (3), the deformed configuration of the medium, i.e. the relative displacements along the discontinuity surface. Furthermore, stresses, strains and displacements at any point of the medium can be easily computed using appropriate influence functions (equation (4)). Finally, stress intensity factors at the tips of the discontinuity can be calculated by means of a simple procedure, since it can be demonstrated that the specific interpolation function used (function f defined by equation (5)) is directly related to the stress intensity factors.[9]

In Figure 4 some values computed for the stress intensity factor in the case of an opening crack embedded in an infinite medium are shown, together with the 'correct' theoretical value obtained using Fracture Mechanics. As can be seen a relatively small number of collocation points is sufficient in order to arrive at very accurate results.

The model here illustrated has been tried out in a variety of examples regarding discontinuities embedded in an infinite medium, with various kinds of loading applied; the results obtained have shown excellent agreement with theoretical values where such are available. In Figures 5 and 6 the results obtained in the case of a straight discontinuity in a stressed infinite medium (Figure 1) are shown.

In Figure 5 computed opening displacements are plotted, while in Figure 6 stresses calculated using the algorithm are compared to those predicted by Fracture Mechanics. It can be seen that the algorithm predicts an elliptical shape for the discontinuity under traction, which is correct. Regarding stress computation, the results obtained are practically coincident with theoretical Fracture Mechanics results for points of the medium very near the discontinuity tip; further away the stresses computed through the algorithm remain realistic in that they tend asymptotically to

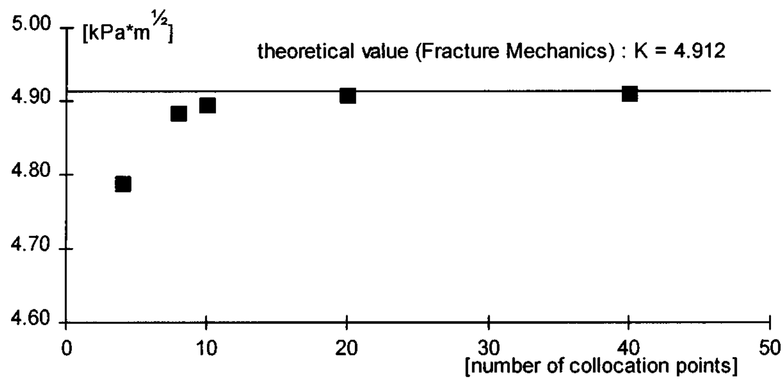

Figure 4. Values computed for the stress intensity factor

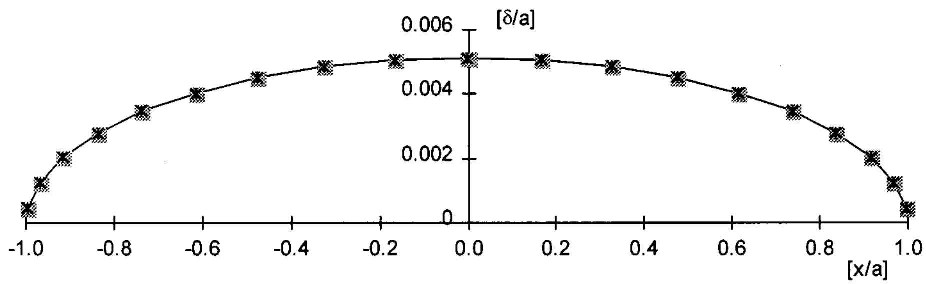

Figure 5. Discontinuity in infinite medium: computed deformed configuration

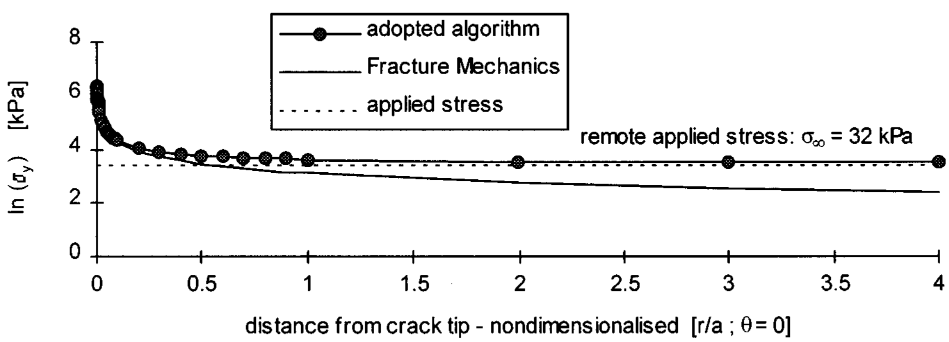

Figure 6. Stress in the vicinity of the tip

the value of the stress applied at infinity, while Fracture Mechanics results are no longer valid; this is due to the fact that the basic equations of Fracture Mechanics (equation (1)) are derived, as already mentioned, by truncation of a series expansion, and therefore only hold for material points that are very near the discontinuity tip (i.e. $r \ll a$). It is important to observe that the distance over which the effects of the presence of the discontinuity are felt can be extremely small, compared to the length of the discontinuity; it should be remembered that, while in the algorithm

here presented this fact arises simply as a result of the computations performed, in Fracture Mechanics this is not a consequence of the particular model adopted but reflects an initial assumption of the theory.

The greatest advantage of the algorithm however is that no limitation needs to be set on the geometry of the discontinuity. This in fact can be straight or curved, and can be composed of two or more branches which can again be straight or curved. It is also possible to consider more than one discontinuity present in the medium, and the influence of each on the other. Furthermore, should a discontinuity surface propagate, or extend its length in some way, the numerical discretization is only slightly affected, and the sole adjustment required is the addition of a new segment at the end of the existing discontinuity.

Having established the reliability of the algorithm for modelling the presence of a discontinuity in an infinite medium, its value for practical applications consists in the possibility of coupling the numerical procedure with a simple finite element discretization technique, thus obtaining a complete model capable of analysing the behaviour of a finite medium in which discontinuities of any kind and of arbitrary geometry are present. In the following paragraph the numerical procedure for coupling the algorithm with the finite element method is presented.

3.3. Displacement discontinuity in a finite body

The algorithm previously described can be coupled to the finite element method procedure by applying the principle of superposition, thus avoiding problems which can arise when using other coupling schemes based on stress- or strain-matching conditions on arbitrarily defined surfaces within the medium.[14,6] Validity of the principle of supersposition in the cases here examined is unambiguous, since all applications are considered within a linear elastic framework, this being in fact a basic assumption of the entire formulation.

Figure 7 illustrates the way in which the principle of superposition is applied. The original problem consists of an elastic body in which one or more discontinuity surfaces are present, these

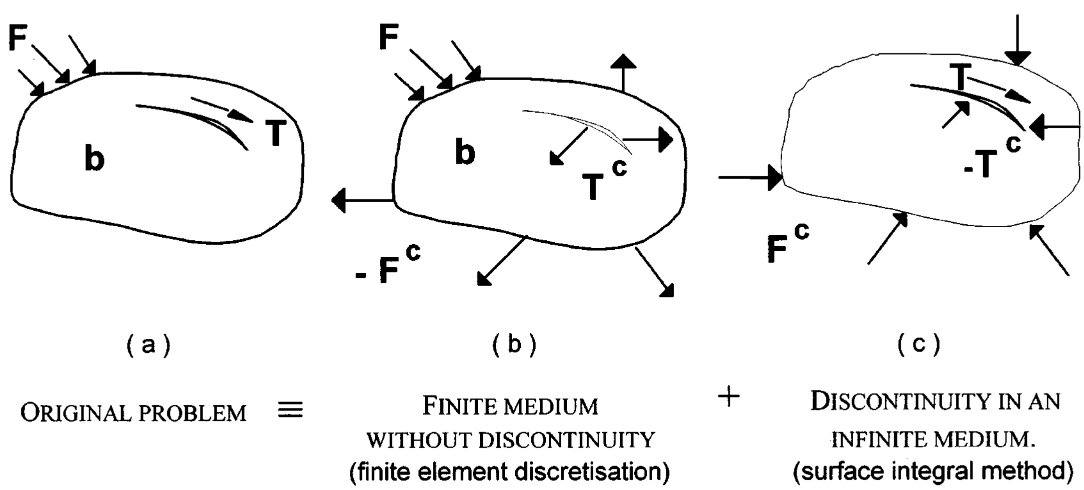

Figure 7. Application of the principle of superposition

latter being either totally embedded in the medium or intersecting its boundary surface; the body can be arbitrarily constrained, and in general will be subjected to a system of body forces **b** and surface tractions **F**. Moreover, a system **T** representing an internal assigned stress state (which can include shear components and internal pressure) may act across the discontinuity surfaces. Using the principle of superposition, this problem can be treated as the sum of two sub-problems, which are, respectively, that of the same finite body without any discontinuity and on which act the previous load systems **b** and **F** plus a fictitious system of tractions $-\mathbf{F}^{C}$, and that of the sole discontinuity in an infinite medium, with the load system **T** acting across its surface plus a system of tractions $-\mathbf{T}^{C}$. These new systems of tractions $\mathbf{T}^{C}$ and $\mathbf{F}^{C}$ are introduced so that the basic requirement of the principle of superposition be satisfied, namely that the sum of the load systems and boundary conditions of the two sub-problems gives the load system and boundary conditions of the actual problem which is being analysed. The physical meaning and computation of these tractions, which depend on the particular coupling technique adopted,[16] are the following:

(1) considering the finite body without the discontinuity, the external loads and boundary conditions will produce a state of stress at the points where, in the complete problem, the discontinuity exists. This state of stress is indicated as $\mathbf{T}^{C}$, and can be calculated as

$$\mathbf{T}^{C} = \mathbf{S} \cdot \mathbf{U}^{FE} \tag{10}$$

where $\mathbf{U}^{FE}$ are the displacements of the FE nodes and **S** is a coupling matrix expressing stress components at a point of the discontinuity surface due to unit displacements of the nodes of the finite element containing that particular point.

(2) considering the sole discontinuity surface embedded in an infinite medium, a state of stress will arise on the points of this medium corresponding to the boundary surface of the original body, as a result of the relative displacements which occur across the discontinuity surface. Recalling the possibility of expressing relative displacements by means of a dislocation density function and using equation (6), this state of stress is indicated by $\mathbf{F}^{C}$ and can be calculated by means of

$$\mathbf{F}^{C} = \mathbf{G}^{*} \cdot f \tag{11}$$

where f is the vector of unknowns in sub-problem (b), related to the values assumed by the dislocation density function at the collocation points along the discontinuity surface and $\mathbf{G}^{*}$ is a coupling matrix expressing the nodal loads that should be applied at the finite element nodes lying along the boundary of the original body so that unit dislocation densities arise at the collocation points along the discontinuity surface. The components of this matrix are computed through numerical discretization of appropriate integrals.

The solutions to the two sub-problems in which the actual problem has been divided can be obtained by applying the finite element method or the algorithm illustrated in the previous paragraph as appropriate. In the matrix form the two equations are

sub-problem (b):
$$\mathbf{K} \cdot \mathbf{U}^{FE} = \mathbf{F} - \mathbf{F}^{C} \tag{12}$$

in which **K** is the stiffness matrix of the finite element mesh (symmetric, banded);

sub-problem (c):
$$\mathbf{C}^{*} \cdot f = \mathbf{T} - \mathbf{T}^{C} \tag{13}$$

in which $\mathbf{C}^{*}$ is the coefficient matrix relative to the integral formulation of the discontinuity problem, obtained through the numerical discretization scheme (non symmetric, full).

In the above equations reference has been made to the vector $\mathbf{U}^{FE}$ indicating displacements of the finite element nodes arising only from the first sub-problem. The total nodal displacements will also take account of the displacements arising as a consequence of the presence of the discontinuity, i.e.

$$\mathbf{U} = \mathbf{U}^{FE} + \mathbf{U}^{SI} \tag{14}$$

The displacement field $\mathbf{U}^{SI}$ can be calculated once the dislocation density functions along the discontinuity are known, by means of

$$\mathbf{U}^{SI} = \mathbf{L} \cdot f \tag{15}$$

where $\mathbf{L}$ is a matrix whose components derive from appropriate influence functions and represent nodal displacements of the finite element mesh due to unit dislocation densities at the collocation points along the discontinuity surface. Combination of equations (10)–(15) leads to the final coupled matrix equation

$$\begin{bmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{S} & \mathbf{C} \end{bmatrix} * \begin{Bmatrix} \mathbf{U} \\ f \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{T} \end{Bmatrix} \tag{16}$$

where $\mathbf{G} = \mathbf{G}^* - \mathbf{KL}$ and $\mathbf{C} = \mathbf{C}^* - \mathbf{SL}$. (17)

Solution of the above algebraic system yields nodal displacements of the finite elements and dislocation density values at collocation points along the discontinuity surface. The nodal displacements thus computed take into account the presence of the discontinuity surface.

3.4. Discussion

The matrix to be inverted is neither symmetric nor banded. However, the additional computational effort required as compared to the usual finite element method matrix inversion is by far compensated by the great simplifications that can be introduced into the finite element discretization, once the discontinuity is separately modelled in the manner above illustrated. In particular, the number of elements required for the finite element mesh can be kept within acceptable limits, since no refined discretization is required in the vicinity of the discontinuity or its tip. In fact, the excessive stress gradients that arise within the (very small) zone surrounding the tip do not have to be captured by the finite element mesh. Furthermore, there is no necessity to modify the finite element mesh even when there is a change in the geometry of the discontinuity. In fact, the behaviour of the discontinuity has been modelled using a surface integral approach based on a discretization of the discontinuity itself which does not in any way depend on the finite element discretization; it follows that the discontinuity can change position, or even propagate 'through' a finite element, without any need to alter the discretization of the body.

An important feature of the method is the possibility to use a very simple linear elastic constitutive law in conjunction with the finite element mesh. This is not a mere approximation introduced for the sake of simplicity, but should be seen as a basic requirement of the method, related to the possibility of coupling a finite element discretization to a surface integral analysis of the discontinuity. In other words, the assumption of a linear elastic constitutive law actually does reflect real behaviour, as can be seen by observation of the global behaviour of a mass containing discontinuities. In this case in fact, even if the constitutive law of the material itself is certainly not

linear elastic, as can be seen through sophisticated laboratory testing, the presence of discontinuities often appears to influence global behaviour so strongly that the material's actual constitutive law may only play a secondary role;[17] in fact, observed behaviour of soil masses confirms that large strains are concentrated around the discontinuity, while the rest of the medium remains practically undeformed and hence can be considered without error within the linear elastic range. In the proposed algorithm therefore the use of linear elasticity for the medium does not imply an assumption regarding the intrinsic behaviour of the soil, but rather an investigation into the possibility to assume this law when considered in conjunction with modelling global behaviour by attributing a central role to the discontinuity.

Regarding the size of the algebraic system to be solved, it should be noted that this is essentially determined by the size of the finite element mesh, since in the system matrix (equation (16)) submatrix $\mathbf{C}$ is very much smaller in size than matrix $\mathbf{K}$. (Taking the full-scale application presented herewith as a typical example, these dimensions are: $K = 500 \times 500$, $C = 40 \times 40$). It can be said that the algorithm consists of adding to a very crude finite element discretization of the problem a system of submatrices which is small but capable of biasing the solution towards the correct one.

It should be observed that the algorithm here presented was initially developed at the M.I.T.; in fact, the computer program here used was derived from the existing code MULTIFRAC, developed at the M.I.T. in relation to investigation of oil extraction by hydraulic fracturing of non-homogeneous rock layers containing inclusions or bimaterial surfaces. Although much of the analytical and numerical procedures are similar, the underlying purpose in the present case is different. In fact, in the original formulation the approach is from a Fracture Mechanics point of view, where it is of interest to model critical conditions and consequent propagation; much more effort on the modelling of the discontinuity and its behaviour is therefore required. In the present case on the other hand procedures are used in order to enhace correct numerical analysis of the finite medium, with particular concern on soil mechanics problems. In fact, global behaviour of the medium as influenced by the presence of discontinuities is of primary interest, and such questions have been addressed as the transmission of compressive and shear stresses across the discontinuity surfaces, calculation of stress and displacement fields in the medium and influence of the geometry and collocation of the discontinuity on these.

3.5. Example

An application of the algorithm to a simple problem is presented, in order to illustrate certain aspects and advantages of the method. The finite element mesh adopted is shown in Figure 8; eight-node quadrilateral quadratic elements with 3×3 numerical integration were used throughout. All analyses assume plane strain conditions. Material constants are the following: shear modulus $G = 6000$ kPa, Poisson's ratio $\nu = 0{\cdot}02$; these values do not correctly represent any soil sample, but were chosen here so that the results obtained for the displacement field privilege the effect of the discontinuity rather than the material's intrinsic stress–strain law. Two aspects of the model are illustrated, namely: (1) the dependence of the K_{II} stress intensity factor on the stress state acting across the discontinuity surface and (2) modelling of the interaction between the boundary surface of a body and an embedded discontinuity.

In the first case, a horizontal discontinuity 1·5 cm long which intersects the external boundary has been considered, as shown in Figure 9. Deformed configurations relative to four loading conditions, together with the stress intensity factor that was calculated, are shown in Figure 10.

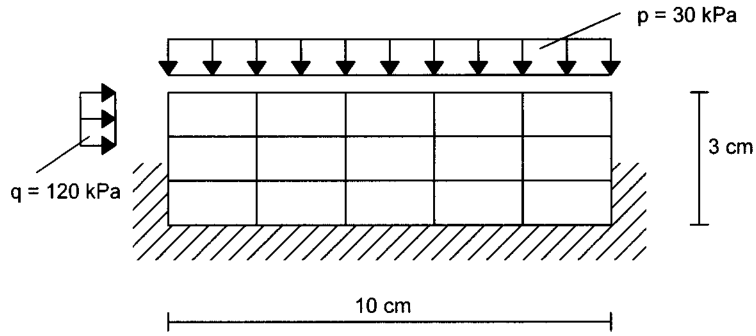

Figure 8. Example of application: finite element mesh and loads

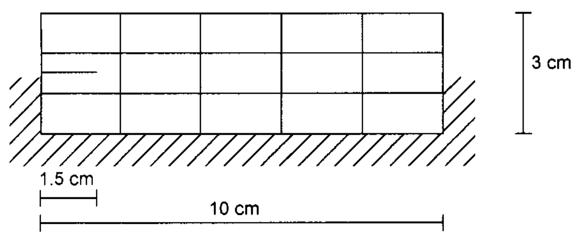

Figure 9. Example of application: discontinuity intersecting external boundary

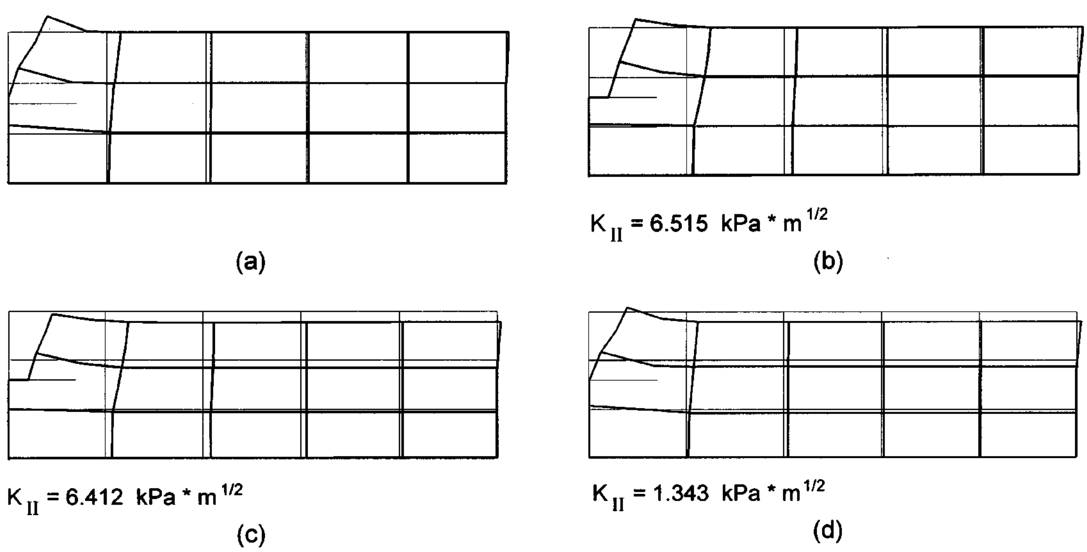

Figure 10. Deformed mesh

These values for K_{II} were calculated as previously illustrated, and represent the strength of the shear stress singularity at the discontinuity tip. The loading conditions are: (a) only horizontal loading $q = 120$ kPa (cf. Figure 8), with no discontinuity surface present in the medium; (b) same loading as in case (a), discontinuity present with zero residual shear strength; (c) both normal

compression $p = 30$ kPa and horizontal loading $q = 120$ kPa applied, discontinuity as in case (b); (d) loading as in case (c), with a residual shear strength of 20 kPa acting across the discontinuity.

It is immediately obvious that the deformed state strongly depends on the presence of the discontinuity, and that the stress intensity factor assumes different values in each case. In particular, when a compressive stress is acting across the discontinuity surface (case (c)), K_{II} decreases slightly, while when a shear stress opposing the free slipping of the discontinuity surfaces is also present (case (d)) the stress intensity factor decreases radically, as does the relative displacement (slip). These results could be said to reflect an analogy between the value of stress intensity factor with the tendency of the discontinuity to propagate, seemingly in confirmation of what was predicted by the authors who have used Fracture Mechanics theory in the assessment of discontinuity propagation in natural slopes assuming the so-called Mode II propagation criterion;[1,2] on the other hand, these results depict the dependency of K_{II} on the stress state when discontinuity interfaces are not stress free. In fact, here K_{II} has been calculated directly as the strength of the stress singularity, as defined by Fracture Mechanics, and not using the habitual Fracture Mechanics formulae whose validity in the case of not stress-free discontinuity surfaces should be questioned.

In order to model the interaction between an embedded discontinuity and the medium's boundary, a 1-cm-long slip surface was considered as shown in Figure 11. The loading consists of only $q = 120$ kPa (Figure 8). The relative displacement calculated across the discontinuity surface and the stress intensity factors computed at the two tips are shown in Figure 12. A comparison with Figure 5, in which the same results regarding a discontinuity embedded in an infinite medium are illustrated, shows that in the case of a discontinuity lying near a boundary surface the deformed shape is not exactly an ellipse but is influenced by the vicinity of the boundary. The same influence can be observed in the values computed for the stress intensity factors, from which it is clear that it is the tip lying nearest to the boundary surface that is more critical regarding the probability of propagation.

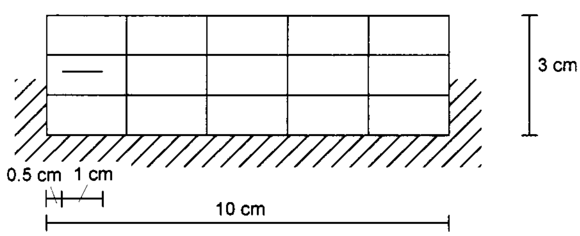

Figure 11. Example of application: embedded discontinuity

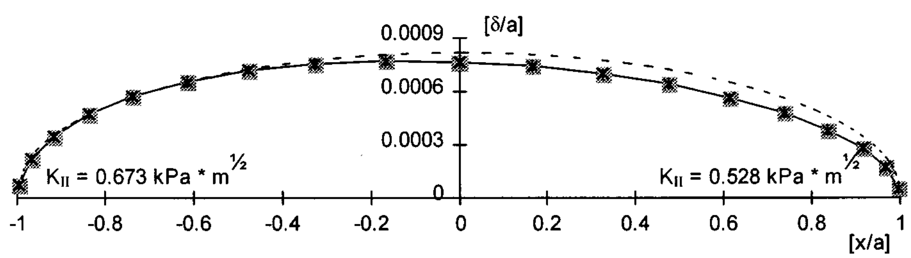

Figure 12. Interaction of discontinuity with boundary: computed deformed configuration

4. APPLICABILITY OF THE MODELS FOR SOIL MECHANICS

There are numerous examples in the literature in which either the Fracture Mechanics approach or the displacement discontinuity model have been applied with the aim to analyse the global behaviour of a soil mass. The first imortant work in this direction is that of Palmer and Rice[1] in which an attempt is made to model the propagation of a shear band in a slope, with some reference to progressive failure. The authors use the Fracture Mechanics approach, in which they assume the existence of a Mode II propagation for shear bands. The results obtained suffer the limitation of having thus postulated *a priori* the direction in which the discontinuity will propagate; furthermore, in order to arrive at a quantitative result the authors make a series of assumptions which are severely limiting in relation to the physical reality of the problem.

Operating within the same frame of ideas Saada et al.,[2] still postulating the possibility of a pure shear mode of propagation, propose a method for determining K_{IIC} through a laboratory experiment; they obtain a linear dependency of such parameter upon the overconsolidation ratio OCR. However, in this case also the same objections can be expressed as in the work of Palmer and Rice; in other words the parameter K_{IIC} determined by means of the proposed procedures will be useful for the evaluation of the stability of a given configuration only in those cases where a slip discontinuity propagates in a self-similar manner (i.e. in a direction parallel to the initial one of the slip band).

Vallejo[3] in a laboratory research project shows that this in fact is never the case. In particular, by means of a detailed parametric study, it is shown how it is possible to separate the observations into two different groups, in which the physical phenomenon involved is collapse through yield in one case, and fracture collapse in the other. Only in this last case does the use of Fracture Mechanics, and in particular of a mixed-mode propagation criterion (maximum principal stress[18]), lead to a coherent mathematical interpretation of the observations. This criterion in fact, which can also be expressed in terms of the values of the two stress intensity factors K_I and K_{II}, assumes that propagation will occur in a direction perpendicular to that along which normal traction is maximum, and thus correctly interprets the physical phenomenon of fracturing as a breaking of the material occurring as the result of a separation or opening at the grain scale.

Both collapse through yield and through fracturing may develop for a particular soil, and the choice of which of the two phenomena prevails will depend on intrinsic material parameters, but also on some suitable state variable. Such state variables could be the past stress or strain history, the overconsolidation ratio, etc. In one specific research project for example, Vallejo[4] shows that the water content w% of a clay can be taken as a significant parameter in order to determine which of the two phenomena will occur: for high values of w% the material behaves in a ductile fashion, and plastic collapse (yield) ensues, while for low values of w% a fragile behaviour is evident and a fracture propagates in the material, in a direction which can be accurately predicted by the above-mentioned maximum principal stress criterion.

The application of the dislocation density method for modelling a soil containing discontinuities can be seen as a numerical approach aiming to analyse the manner in which the presence of these discontinuities is reflected upon the global stress–strain behaviour of the medium. On the other hand, this method does not give any indication regarding the possibility of propagation of the discontinuity; in fact, this is related to the response offered by the medium to a stress state, or in other words to its strength capacity which may or may not be sufficient to contrast a propagation phenomenon due to that stress state; such phenomenon may be either a rupture due to

separation at microscale, or plastic collapse due to the overcoming of the material's shear strength. In the cases where the phenomenon which ensues is of the first type, the criterion based upon the stress intensity factors is appropriate and can still be used.

5. A FULL-SCALE APPLICATION

A full-scale application of the model regarding an excavation in a clayey soil is presented. The basic geometry for this example was taken from an excavation actually performed in Chicago and which has been monitored and presented in the literature,[19,20] and for which a great amount of experimental data and extensive information is available; in particular, measurements indicate the presence and location of zones of shear localization, which can be modelled as discontinuity surfaces. The excavation was performed in various stages, and soil behaviour was monitored throughout.

The soil profile comprises various strata, which consist of lightly overconsolidated clays from ground surface down to about 15 m, after which a very stiff overconsolidated clay is encountered. In more detail, these lightly OC clays are a soft clay down to 9 m, over a medium stiff clay extending to 15 m. The geotechnical characterization of these soils gave values of undrained shear strength which vary linearly with depth.

In Figure 13 the infinite element mesh used is shown. Eight-node quadrilateral isoparametric elements with 3×3 numerical integration were used, and plane strain analysis was performed. The model requires elastic parameters E (Young's modulus) and v (Poisson ratio) to be assigned; these were chosen in such a way as to remain as much as possible coherent to values determined through measurements. A Young's modulus for undrained conditions was therefore arrived at by multiplying the undrained shear strength by 300, which is typical of lightly overconsolidated

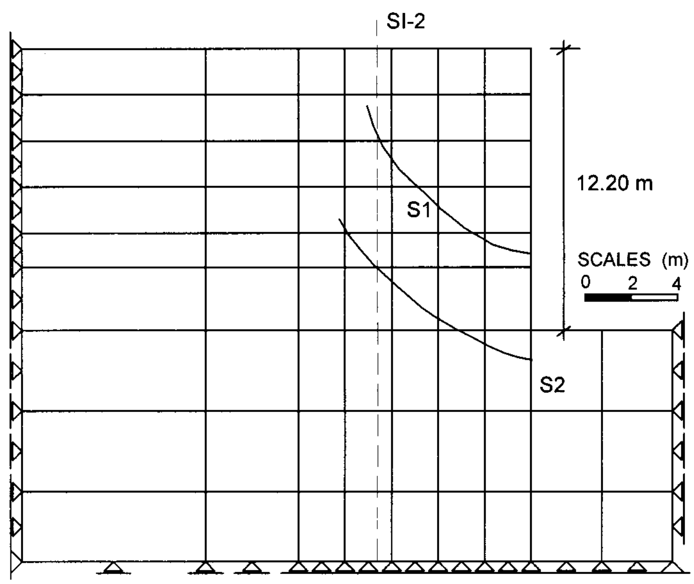

Figure 13. Finite element mesh (excavation to 12·20 m) and position of discontinuities

clays. The mean value of undrained strength c_u in the clay layers is 23·3 kPa. Therefore, a Young's modulus E equal to 300*23·3 $\cong$ 7000 kPa was assumed. Poisson's ratio was taken equal to 0·5, to correctly describe undrained analysis in saturated soils (condition of zero volumetric strain).

The most important feature of available data regarding the Chicago site is the presence of inclinometer and extensometer data which clearly indicate the formation of two shear bands within the soil mass, and their exact collocation.[21] These were therefore introduced in their actual geometrical position within the finite element mesh, as shown in Figure 13. Each discontinuity surface is modelled as a series of linear segments, and 20 collocation points are considered; these are concentrated near the tips of each discontinuity. In reality the shear bands did not appear simultaneously, since available data refer to various situations which were recorded as excavation operations proceeded. The model is here applied to analyse two different configurations, namely:
(1) excavation to 9·50 m (H950), and
(2) excavation to 12·20 m (H1220).

In both cases the analysis was performed twice, once with only the first discontinuity present (S1), and once with both (S1S2). Computation results are shown in Figures 14–18. Figure 14 shows the deformed mesh, in which computed displacement components across the discontinuity are included.

In Figures 15 and 16 computed ground surface settlements are compared to measured values. It should be pointed out that vertical displacements are computed as arising from two separate effects, namely the soil's self-weight, by which displacements depend on the size of the mesh and the values of elastic parameters assumed, and the additional geometric constraint that arises by allowing the possibility of sliding along the discontinuity.

Therefore, the numerical values of computed displacements are not particularly important, even though an apparent excellent agreement of these with measured values may seem evident.

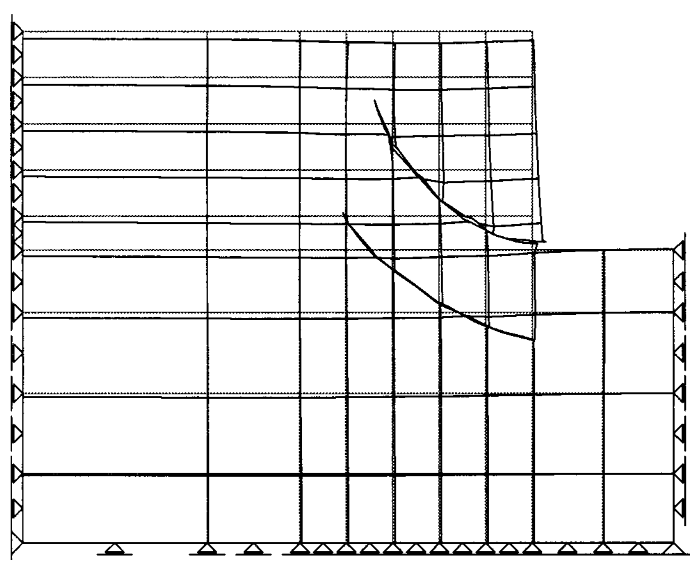

Figure 14. Deformed mesh; excavation to 9·50 m

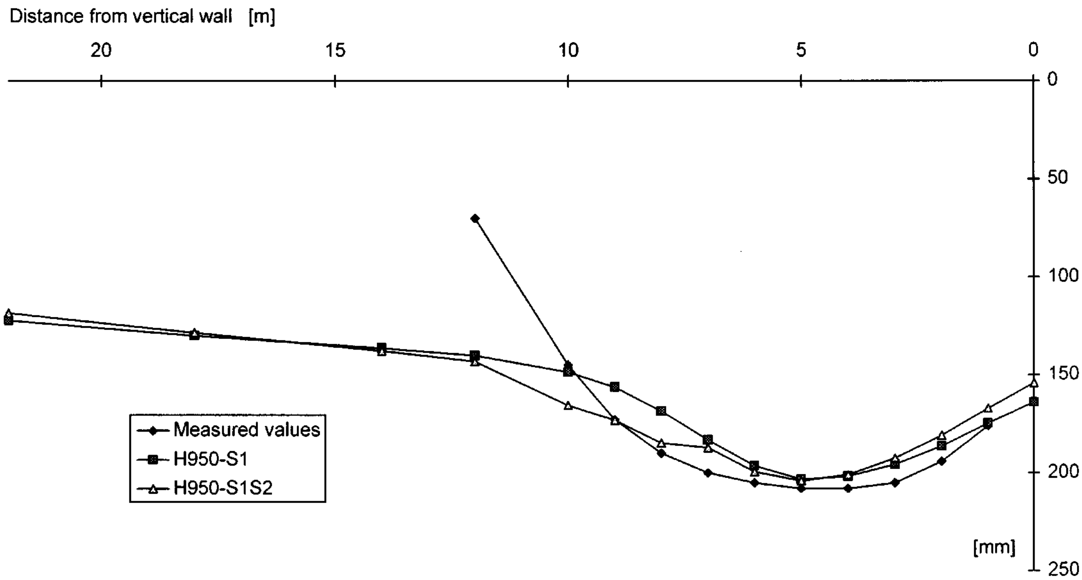

Figure 15. Ground surface settlements: computed and measured values; excavation to 9·50 m

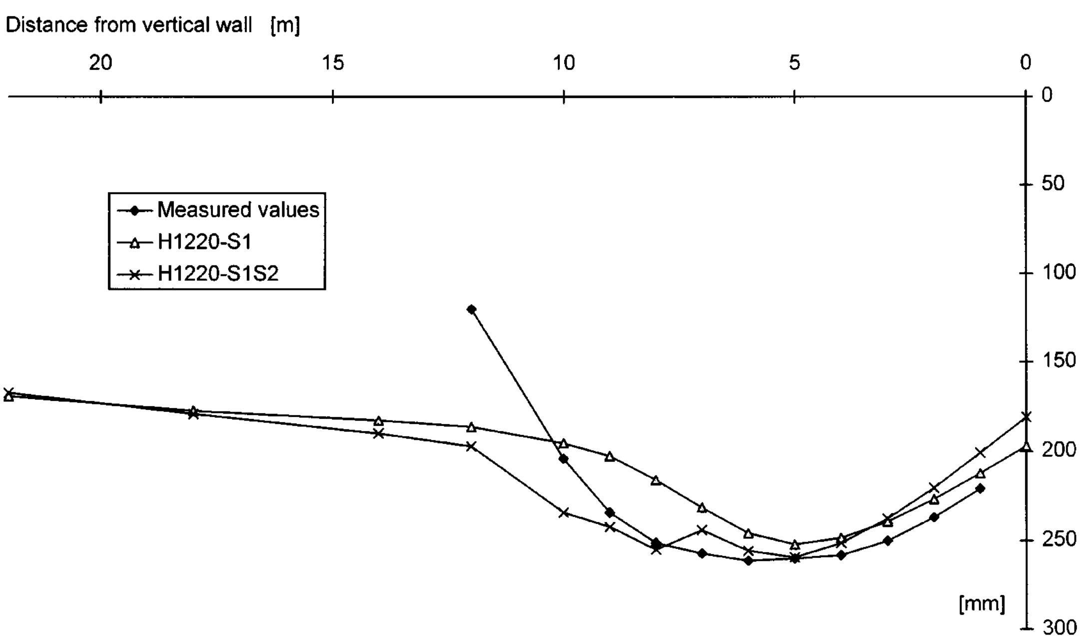

Figure 16. Ground surface settlements: computed and measured values; excavation to 12·20 m

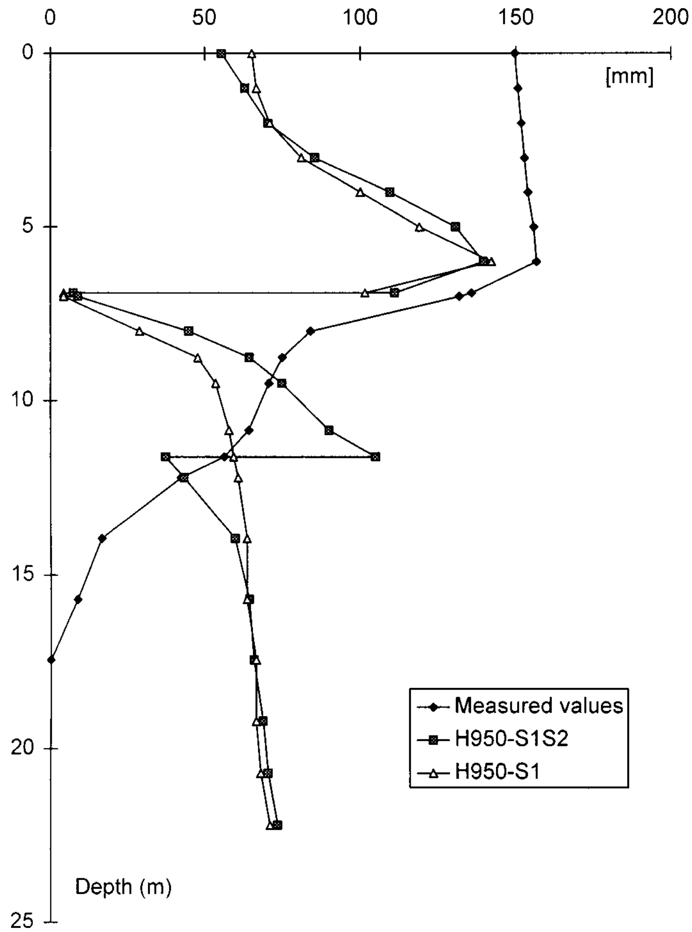

Figure 17. Horizontal displacements: computed and measured values; excavation to 9·50 m

On the other hand, the fact that the shape of the displacement profile closely matches that of the measured settlements is highly significant, especially as this is true in the very critical zone just behind the excavation.

In Figures 17 and 18 computed horizontal displacements at appropriate finite element nodes are compared to those measured by inclinometer SI-2 (see Figure 13). Good quantitative agreement is again evident, down to a depth of about 9 m. Further down the trend differs, which could be explained by the oversimplification of the finite element mesh, in which constant soil properties are assumed throughout; in fact the real soil becomes stiffer at that depth. The same explanation holds for the topmost layer, where computed displacements are much smaller than measured ones; this layer in fact has been modelled as a clay, while in reality it consists of a top fill layer, which is evidently carried horizontally by the underlying soil.

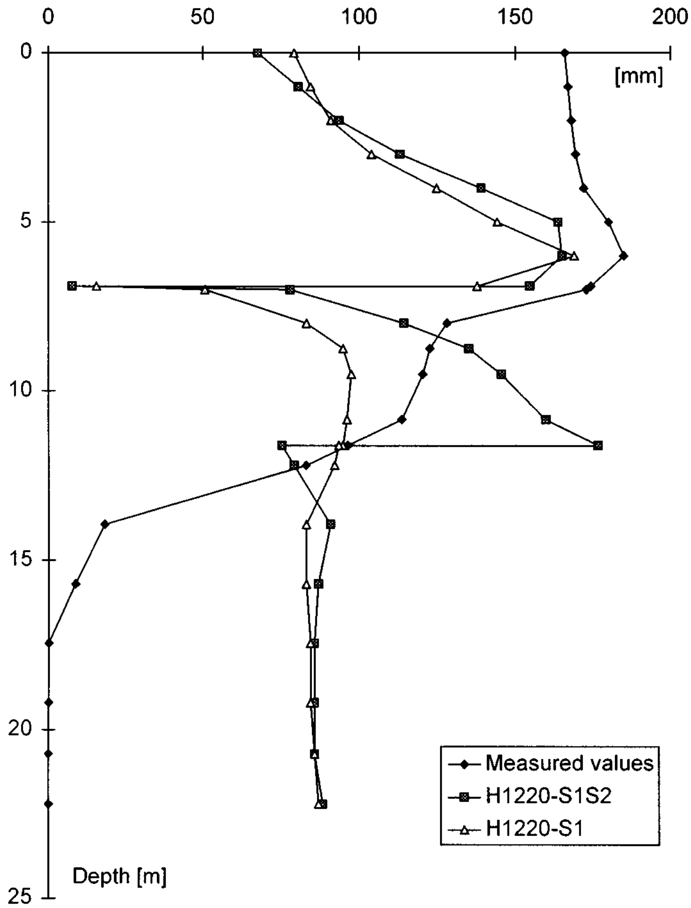

Figure 18. Horizontal displacements: computed and measured values; excavation to 12·20 m

6. CONCLUSIONS

A numerical model for evaluating the behaviour of a medium containing discontinuities has been presented. In principle the model can handle any number of discontinuities, which can be of arbitrary shape and collocated anywhere in the medium; in the case of curved discontinuity surfaces, these are analysed as the union of small linear segments. Various internal stress situations can be taken into account (for example shear stresses, compression), and it is possible to allow for the transmission of compressive stresses across a discontinuity surface which would thus behave as a closed crack with the discontinuity surfaces sliding against one another. Only two constants are needed to describe the material, as in linear elastic theory. The model also predicts correctly the interaction of a discontinuity with the boundary of the medium, and can take into account discontinuities which actually intersect the boundary or originate at a boundary point. The model is based on the combination of two separate formulations, and in particular a finite element discretization which uses a simple linear elastic constitutive law for analysing the soil

mass, and a surface integral method for analysing a discontinuity. The latter one is based on concepts of the theory of dislocations, in order to represent the discontinuity as a mathematically equivalent continuous distribution of point dislocations. The two models are coupled together using the principle of superposition, thus obtaining a global solution in which the behaviour of the soil mass and that of the discontinuity are treated in separate ways but in such a fashion that the one takes the other correctly into account.

The model introduces some important advantages over other numerical methods available for the analysis of similar problems. Namely, many of the difficulties which are frequently encountered in the discretization of large bodies or soil masses containing discontinuities are avoided, because the modelling of a discontinuity, its behaviour and its interaction with the continuous medium are performed separately. This facilitates not only geometrical discretization, but also the constitutive modelling of the material itself. In fact, it is assumed that the particular displacement configuration which arises in bodies containing discontinuities depends mostly on the discontinuities themselves, rather than on the material's intrinsic constitutive behaviour. It follows that when the behaviour of the discontinuity is correctly modelled, it is possible to assume a simple linear elastic law for the material.

The analytical equations by which the discontinuity behaviour is modelled are solved by means of a particular numerical scheme, which yields a solution to the problem that is mathematically exact. The proposed model is also advantageous from a numerical–computational point of view, since a change in the geometrical configuration of the discontinuity does not require any variation of the finite element mesh.

It is however pointed out that, although the formulation of the algorithm makes it extremely flexible and suitable for use in various contexts, it is currently being applied only for the deformation analysis of a continuum containing discontinuity surfaces, in which the interaction of these surfaces with each other and with the boundary is taken into account. No analysis regarding the propagation or the initiation of discontinuities is attempted; to that end a greater insight into the material's intrinsic behaviour is required.

A drawback of the method is the necessity of the presence of some kind of discontinuity to start with, even as an initial notch; stability analysis would seem necessary in order to model initiation of shear bands and overcome this limitation. However, it is also worth mentioning that in most geotechnical applications it is the nature of the structure itself that suggests the most probable location for shear band initiation.

Other possible modelling approaches for the same kind of situation based directly on Fracture Mechanics are briefly discussed. It is shown that these are in fact applicable only under specific restrictions, while very limited information as regards global soil behaviour can be derived through their use.

Some simple working examples of the displacement discontinuity model are provided, which aim at illustrating a few particular points and some advantages that arise in its use. Finally, a full-scale application of the model in the case of an excavation problem is presented. This confirms the possibility to arrive at a satisfactory prediction of displacements both in a local and in a global sense.

The proposed model can be therefore considered as a powerful tool for the analysis of actual problems, or for parametric analysis in research projects where the influence of discontinuities on overall behaviour is examined, as in fissured clays and in all materials where overall behaviour is determined not so much by the constitutive properties of the material itself, but rather by the presence of discontinuities.

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